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# On the distribution of certain Pisot numbers

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## Abstract

A Pisot number  $\theta$  is said to be simple if the beta-expansion of its fractional part, in base  $\theta$ , is finite. Let  $\mathbb{P}$  be the set of such numbers, and  $\mathbb{S} \setminus \mathbb{P}$  be the complement of  $\mathbb{P}$  in the set  $\mathbb{S}$  of Pisot numbers. We show several results about the derived sets of  $\mathbb{P}$  and of  $\mathbb{S} \setminus \mathbb{P}$ . A Pisot number  $\theta$ , with degree greater than 1, is said to be strong, if it has a proper real positive conjugate which is greater than the modulus of the remaining conjugates of  $\theta$ . The set, say  $\mathbb{X}$ , of such numbers has been defined by Boyd (1993) [5], and is contained in  $\mathbb{S} \setminus \mathbb{P}$ . We also prove that the infimum of the  $j$ -th derived set of  $\mathbb{X}$ , where  $j$  runs through the set of positive rational integers, is at most  $j + 2$ .

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## 1. Introduction

For a real number  $\beta > 1$ , let  $(r_n(\beta), \varepsilon_n(\beta))_{n \geq 1} = (r_n, \varepsilon_n)_{n \geq 1}$  be the sequence defined by the relations

$$(r_n, \varepsilon_n) := (\{\beta r_{n-1}\}, [\beta r_{n-1}]),$$

where  $\{ \}$  and  $[ \ ]$  are, respectively, the fractional and integer part functions,  $r_0 = r_0(\beta) := \{\beta\}$ , and  $n$  runs through the set  $\mathbb{N}$  of positive rational integers. Then,  $r_n \in [0, 1[$ ,  $\varepsilon_n \in$

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$\{0, 1, \dots, \max_{k \in \mathbb{N} \cap [1, \beta[} k\}, \{\beta\} = \sum_{n \geq 1} \varepsilon_n / \beta^n$ , and we write

$$\{\beta\} \equiv \sum_{n \geq 1} \frac{\varepsilon_n}{\beta^n}.$$

The sequence  $(\varepsilon_n)_{n \geq 1}$ , called beta-expansion of  $\{\beta\}$ , was defined, in a more general context, by Rényi [17]. If this sequence is periodic, then  $\beta$  is said to be a beta-number; in particular, when  $(\varepsilon_n)_{n \geq 1}$  terminates only by zeros, the number  $\beta$  is called a simple beta-number. Beta-numbers have been defined and studied by Parry [15]. One of the results of [15], asserts that beta-numbers are algebraic integers, the conjugates of a beta-number  $\beta$ , are of modulus at most  $\min(2, \beta)$ , and simple beta-numbers are dense in the interval  $[1, \infty[$ .

An easy argument shows that Pisot numbers are beta-numbers [19]. A Pisot number is a real algebraic integer greater than 1, whose other conjugates are of modulus less than 1. Some results related to the beta-expansion of real numbers, in a base which is a Pisot number, may be found in [1,6,10,24]. A large amount of the structure of the set, usually denoted as  $\mathbb{S}$ , of Pisot numbers is understood. For example,  $\mathbb{S}$  is closed in the real line  $\mathbb{R}$  [18],  $\min \mathbb{S} = \theta_0 = 1.3247\dots$ , where  $\theta_0^3 - \theta_0 - 1 = 0$  [20], the elements of  $\mathbb{S} \cap ]1, 1.6183\dots[$  are known [8] (see also Theorem 7.2.1 of [4], or the proof of Theorem 1 below),  $\min \mathbb{S}^{(1)} = \frac{1+\sqrt{5}}{2} = 1.6180\dots$  [8], a complete list of the elements of  $\mathbb{S}^{(1)} \cap ]1, 2[$  is exhibited in [2] (see also Theorem 1.6 of [14]), and  $\min \mathbb{S}^{(2)} = 2$  [11]. Throughout, the notation  $I^{(j)}$ , where  $I \subset \mathbb{R}$  and  $j \in \mathbb{N}$ , means the  $j$ -th derived set of  $I$ .

The distribution of the elements of the set, say  $\mathbb{P}$ , of Pisot numbers which are simple beta-numbers, is not completely known. In a very recent paper, Panju [14] determined the beta-expansion of  $\{\beta\}$ , when  $\beta$  runs through the set of regular Pisot numbers in the interval  $]1, 2[$ . A regular Pisot number less than 2, is an element of  $\mathbb{S}^{(1)} \cap ]1, 2[$ , or is a term of a sequence that tend to an element of  $\mathbb{S}^{(1)} \cap ]1, 2[$ . Hence, for any  $\varepsilon > 0$ , there are at most finitely many non-regular Pisot numbers in  $[1, 2 - \varepsilon]$ . The sequences of Pisot numbers that approach any element of  $\mathbb{S}^{(1)} \cap ]1, 2[$  are completely understood [23].

The following theorem collects some results about the structures of the set  $\mathbb{P}$  and of its complement  $\mathbb{S} \setminus \mathbb{P}$  in  $\mathbb{S}$ . The first and second parts are essentially consequences of the computation done in [14].

**Theorem 1.** *With the notation above, we have:*

- (i)  $\mathbb{P} \cap ]1, (1 + \sqrt{5})/2] = \mathbb{S} \cap ]1, (1 + \sqrt{5})/2]$ ;
- (ii)  $\inf \mathbb{P}^{(2)} = \inf(\mathbb{S} \setminus \mathbb{P})^{(2)} = 2$ , and the two sets  $\mathbb{P}$  and  $\mathbb{S} \setminus \mathbb{P}$  are not closed;
- (iii) if  $a \in \mathbb{N} \cap [2, \infty[$ , then  $a \in \mathbb{P}^{(2)}$ .

It is clear that beta-numbers of degree 1, belong to  $\mathbb{P}$ , and a short computation shows that quadratic beta-numbers, are also Pisot numbers (see the proof of Theorem 2(i) below). In [3], Bassino has proved that simple cubic beta-numbers are Pisot numbers too. Next, we complete these results by proving the following theorem.

**Theorem 2.** *The following propositions are true.*

- (i) Let  $\beta$  be a beta-number, with degree  $d$ . Then,  $\beta$  is a Pisot number, when  $d \leq 2$ , or when  $\beta$  is simple and  $d = 3$ .
- (ii) There are infinitely many cubic beta-numbers which are not Pisot numbers.
- (iii) For any  $d \in \mathbb{N} \cap [4, \infty[$ , there are infinitely many simple beta-numbers, with degree  $d$ , which are not Pisot numbers.

A Pisot number  $\theta$  with degree  $d$  is said to be strong, if  $d = 1$ , or if  $d \geq 2$  and  $\theta$  has a proper real positive conjugate, which is greater than the absolute value of the  $d - 2$  remaining conjugates of  $\theta$ . The set, say  $\mathbb{X}$ , of strong Pisot numbers has been defined by Boyd [5], and its elements appear in a paper of Dubickas [7] as examples of  $Z$ -numbers. A theorem of Pisot [16], asserts that a real number  $\beta > 1$ , is a Pisot number, if and only if, there is  $\lambda \in \mathbb{R} \setminus \{0\}$ , such that  $\sum_{n \in \mathbb{N}} \|\lambda \beta^n\|^2 < \infty$ , where  $\|t\| = \min\{\{t\}, 1 - \{t\}\}$  is the usual distance from  $t$  to the ring of rational integers  $\mathbb{Z}$ . Using this result of Pisot, the author obtained, recently [27], a characterization of the elements of  $\mathbb{X}$ , among real numbers greater than 1:

$$\beta \in \mathbb{X} \iff \forall \varepsilon > 0, \quad \exists \lambda \in \mathbb{R} \setminus \{0\} \mid \sum_{n \geq 0} \{\lambda \beta^n\}^2 < \varepsilon.$$

It is worth noting, that the relation above, does not allow to prove that  $\mathbb{X}$  is a closed subset of  $\mathbb{R}$ , in a similar way that it has been done by Salem, for the set  $\mathbb{S}$  [18]. I am not able to prove (or disprove) that  $\mathbb{X}$  is closed. The following theorem gives explicitly some elements of  $\mathbb{X}^{(j)}$ , where  $j$  runs through  $\mathbb{N}$ .

**Theorem 3.** For  $a \in \mathbb{N} \cap [3, \infty[$ ,  $k \in \{1, \dots, a - 2\}$ , and  $\{n_1, \dots, n_k\} \subset \mathbb{N}$ , let  $P_{(n_k, \dots, n_0; a)}$  be the polynomial defined by the relation

$$P_{(n_k, \dots, n_0; a)}(x) = x^{n_k} P_{(n_{k-1}, \dots, n_0; a)}(x) + 1, \quad (1)$$

where  $P_{(n_0; a)}(x) := x - a$ . Then,  $P_{(n_k, \dots, n_0; a)}$  is the minimal polynomial of a strong Pisot number  $\theta_{(n_k, \dots, n_0; a)} \in ]a - 1, a]$ , and  $\lim_{n_k \rightarrow \infty} \theta_{(n_k, \dots, n_0; a)} = \theta_{(n_{k-1}, \dots, n_0; a)}$ .

A proposition in [27], asserts that  $\mathbb{N} \cap [3, \infty[ \subset \mathbb{X}^{(1)}$ . Theorem 3 improves this proposition.

**Corollary.** With the notation of Theorem 3,  $\theta_{(n_k, \dots, n_0; a)} \in \mathbb{X}^{(a-2-k)}$ .

It follows immediately that  $\theta_{(n_0; a)} = a \in \mathbb{X}^{(a-2)}$ , and so  $\inf \mathbb{X}^{(j)} \leq j + 2$ , for all  $j \in \mathbb{N}$ . Notice also that the corollary above completes Theorem 1(iii), since a strong Pisot number  $\theta$ , with degree greater than 1, has a real positive conjugate, and so  $\theta \in \mathbb{S} \setminus \mathbb{P}$  (it is easy to see from the proof of Theorem 3, when  $k \geq 1$ , that  $\theta_{(n_k, \dots, n_0; a)} \in ]a - 1, a]$ , and so  $\theta_{(n_k, \dots, n_0; a)}$  is not of degree 1).

A Pisot number whose other conjugates are of modulus less than  $\varepsilon$ , where  $\varepsilon$  is fixed in the interval  $]0, 1]$ , is called an  $\varepsilon$ -Pisot number [9]. Let  $K$  be a real algebraic number field. Then, a result of Fan and Schmeling [9] asserts that the set of  $\varepsilon$ -Pisot numbers generating  $K$ , is relatively dense in  $[1, \infty[$ , that is there is a positive constant  $\rho$ , depending only on  $\varepsilon$  and  $K$ , such that each subinterval of  $[1, \infty[$ , of length  $\rho$ , contains an  $\varepsilon$ -Pisot numbers generating  $K$ . We may ask the same question for strong Pisot numbers, when the field  $K$  has a proper real conjugate, and in particular when  $K$  is generated by a Salem number. Recall that a Salem number is a real algebraic integer greater than 1 whose other conjugates are of modulus at most 1 and with a conjugate of modulus 1. Using some results of Meyer on harmonious sets [12], we obtain the following theorem.

**Theorem 4.** Let  $K$  be a real algebraic number field, with degree  $d$ , having at least one proper real conjugate when  $d \geq 2$ , and let  $\varepsilon \in ]0, 1]$ . Then, the following assertions are true.

- (i) The set of  $\varepsilon$ -strong Pisot numbers generating  $K$ , is relatively dense in  $[1, \infty[$ .
- (ii) If  $K$  is generated by a Salem number, then there is a finite subset, say  $F = F(K, \varepsilon)$ , of the integers of  $K$  such that each Salem number generating  $K$  can be written as a sum of an element of  $F$  and an  $\varepsilon$ -strong Pisot number with degree  $d$ .

In this paper, when we speak about a conjugate, the degree or the minimal polynomial of an algebraic integer, we mean over the field of the rationals  $\mathbb{Q}$ . Irreducible polynomials or the degree of a number field are considered over  $\mathbb{Q}$  too. The proof of [Theorem 2](#) uses elementary statements, and the one of [Theorem 1](#) is mainly based on some results of [14]. To simplify the proof of [Theorem 3](#), we shall use a theorem of Smyth [21], which says that two conjugates of a Pisot number having the same modulus are necessary complex conjugates. An easy application of an argument, due to Meyer, gives [Theorem 4\(i\)](#). Finally, notice that the proof of [Theorem 4\(ii\)](#) is similar to the one of [Theorem 2](#) of [25], with minor modifications; for convenience of the reader, we give some details of this proof.

## 2. The proofs

**Proof of Theorem 1.** (i) In [8], Dufresnoy and Pisot have shown that the minimal polynomial of a Pisot number, say  $\theta$ , less than  $\theta_\infty := (1 + \sqrt{5})/2$ , is one of the following polynomials:  $A(x) = x^6 - 2x^5 + x^4 - x^2 + x - 1$ ,  $B_{2k}(x) = (x^{2k}(x^2 - x - 1) + 1)/(x - 1)$ ,  $B_{2k+1}(x) = (x^{2k+1}(x^2 - x - 1) + 1)/(x^2 - 1)$ , or  $C_k(x) = x^k(x^2 - x - 1) + (x^2 - 1)$ , where  $k$  runs through  $\mathbb{N}$ . Hence, the root  $\theta = 1.5617\dots$  of the polynomial  $A$  is the only non-regular Pisot number in  $[1, \theta_\infty]$ . A direct calculation gives for this last case,  $r_0 = \theta - 1$ ,  $(r_1, \varepsilon_1) = (\theta^2 - \theta, 0)$ ,  $(r_2, \varepsilon_2) = (\theta^3 - \theta^2 - 1, 1)$ ,  $(r_3, \varepsilon_3) = (\theta^4 - \theta^3 - \theta, 0)$ ,  $(r_4, \varepsilon_4) = (\theta^5 - \theta^4 - \theta^2, 0)$ ,  $(r_5, \varepsilon_5) = (\theta^6 - \theta^5 - \theta^3 - 1, 1)$ ,  $(r_6, \varepsilon_6) = (\theta^7 - \theta^6 - \theta^4 - \theta, 0)$  and  $r_7 = \theta^8 - \theta^7 - \theta^5 - \theta^2 - 1 = (\theta^2 + \theta + 1)A(\theta) = 0$ ; thus

$$\{\theta\} \equiv \frac{1}{\theta^2} + \frac{1}{\theta^5} + \frac{1}{\theta^7}$$

and  $\theta \in \mathbb{P}$ . If  $B_{2k}(\theta) = 0$  (resp., if  $B_{2k+1}(\theta) = 0$ ,  $C_{2k}(\theta) = 0$ ,  $C_{2k+1}(\theta) = 0$ ), then the results of [14] yield  $\{\theta\} \equiv \sum_{j=1}^{k-1} \frac{1}{\theta^{2j}} + \sum_{j=k+1}^{2k} \frac{1}{\theta^{2j}}$  (resp.,  $\{\theta\} \equiv \sum_{j=0}^k \frac{1}{\theta^{2j}}$ ,  $\{\theta\} \equiv \left(\sum_{j=1}^{k-1} \frac{1}{\theta^{2j}}\right) + \frac{1}{\theta^{4k-1}}$ ,  $\{\theta\} \equiv \left(\sum_{j=1}^{k-1} \frac{1}{\theta^{2j}}\right) + \frac{1}{\theta^{2k+1}} + \frac{1}{\theta^{4k+2}}$ ), and so  $\theta \in \mathbb{P}$ . [Theorem 1\(i\)](#) follows immediately, since  $\{\theta_\infty\} = \theta_\infty - 1 \equiv 1/\theta_\infty$ .

(ii) From the result above we have  $\inf \mathbb{P}^{(1)} = (1 + \sqrt{5})/2$ . Let  $\theta_k > 1$  be a root of the polynomial  $x^k(x^2 - x - 1) - (x^2 - 1)$ , where  $k \in \mathbb{N}$ . Then,  $\theta_k \in \mathbb{S} \cap ]\theta_\infty, \infty[$ ,  $\lim_{k \rightarrow \infty} \theta_k = \theta_\infty$  [8], and  $\theta_k \in \mathbb{S} \setminus \mathbb{P}$  [14]; thus

$$\inf \mathbb{S} \setminus \mathbb{P} = \inf(\mathbb{S} \setminus \mathbb{P})^{(1)} = (1 + \sqrt{5})/2.$$

In particular, we see that  $\mathbb{S} \setminus \mathbb{P}$  is not closed. From the proof of [Theorem 1\(iii\)](#), we shall easily deduce that each interval of the form  $[a, a + 1]$ , where  $a \in \mathbb{N} \cap [2, \infty[$ , contains infinitely many elements of the set  $\mathbb{P}^{(1)} \cap \mathbb{S} \setminus \mathbb{P}$ , and from this we obtain the non-closure of  $\mathbb{P}$ . Notice also that the equality  $\inf(\mathbb{S})^{(2)} = 2$ , implies  $\inf(\mathbb{S} \setminus \mathbb{P})^{(2)} \geq 2$  and  $\inf \mathbb{P}^{(2)} \geq 2$ . Now, let  $L_k(x) := x^k - \sum_{j=1}^k x^{k-j}$ , where  $k \in \mathbb{N} \cap [2, \infty[$ . It is well known that  $L_k$  is the minimal polynomial of a Pisot number, say  $l_k$ , satisfying  $\lim_{k \rightarrow \infty} l_k = 2$  and  $l_k \in \mathbb{S}^{(1)} \cap ]1, 2[$  (see [2] or [23]). If we consider the sequence of Pisot numbers  $(\theta_{j,k})_{j \geq k+1}$ , where  $\theta_{j,k}$  is a root of the polynomial

$$E_{j,k}(x) := x^j L_k(x) - (x^{k+1} - 1),$$

then  $\theta_{j,k} \in \mathbb{S} \setminus \mathbb{P}$  [14]. It follows by the equality  $\lim_{j \rightarrow \infty} \theta_{j,k} = l_k$ , that  $\inf(\mathbb{S} \setminus \mathbb{P})^{(2)} \leq 2$  and so  $\inf(\mathbb{S} \setminus \mathbb{P})^{(2)} = 2$  (using the relation  $l_k^k - \sum_{j=1}^k l_k^{k-j} = 0$ , a short computation shows that

$\{l_k\} = l_k - 1 \equiv \sum_{j=1}^{k-1} 1/l_k^j$ , and so  $l_k \in \mathbb{P}$ ; this shows again that  $\mathbb{S} \setminus \mathbb{P}$  is not closed). Similarly as for  $E_{j,k}$ , we obtain that each polynomial of the form

$$x^j L_k(x) - (x^k - 1)/(x - 1),$$

where  $j \geq k + 1$ , has a root, say again  $\theta_{j,k}$ , which is a Pisot number. Moreover, we have  $\theta_{j,k} \in \mathbb{P}$ ,  $\lim_{j \rightarrow \infty} \theta_{j,k} = l_k$  and so  $\inf P^{(2)} = 2$ .

(iii) Let  $a \in \mathbb{N} \cap [2, \infty[$ . Then, the polynomial

$$P(x) = P_{(k,a)}(x) := x^k(x - (a + 1)) + (a - 1),$$

where  $k \in \mathbb{N}$ , is the minimal polynomial of a Pisot number  $\theta_k = \theta_{(k,a)}$ . Indeed, we have for  $|z| = 1$ ,  $|z^k(z - (a + 1))| = |z - (a + 1)| \geq a > a - 1$ , and so by Rouché's theorem, the polynomial  $P$  has  $k$  roots inside the unit circle, and one root  $\theta_k$  with modulus greater than 1 (the modulus of the product of the roots of  $P$  is  $a - 1 \geq 1$ ). Moreover, the relations  $P(a + 1) = a - 1$ ,  $P(a) = a - a^k - 1$  and  $|\theta_k - (a + 1)| = (a - 1)/\theta_k^k$ , yield  $\theta_k \in ]a, a + 1[ \cap \mathbb{S}$ , and  $\lim_{k \rightarrow \infty} \theta_k = a + 1$ . Now, consider the polynomial

$$M(x) = M_{(n,k,a)}(x) := x^{n-k} P_{(k,a)}(x) + 1,$$

where  $n \in [k + 1, \infty[$ . Then,

$$M(x) = (x - 1) \left( x^n - a \sum_{j=1}^k x^{n-j} - \sum_{j=k+1}^n x^{n-j} \right),$$

and by a result of Perron, cited in [10], the factor  $x^n - a \sum_{j=1}^k x^{n-j} - \sum_{j=k+1}^n x^{n-j}$  of  $M(x)$ , is the minimal polynomial of a Pisot number  $\beta_n = \beta_{(n,k,a)}$ . It is easy to check that  $\beta_n \in \mathbb{P}$  (this result is also a corollary of Theorem 2 of [10]), and the equality  $|P(\beta_n)| = 1/\beta_n^{n-k}$  gives  $\lim_{n \rightarrow \infty} \beta_n = \theta_k$ . It follows that each  $\theta_k$  is a limit of a sequence of elements of  $\mathbb{P}$ , and so  $a + 1 \in \mathbb{P}^{(2)}$ . This ends the proof of Theorem 1(iii), because the relation  $2 \in \mathbb{P}^{(2)}$ , follows from Theorem 1(ii). Finally, notice that  $\theta_k \in \mathbb{S} \setminus \mathbb{P}$ , since  $P(0) = a - 1$  and  $P(1) = -1$ ; thus  $a + 1 \in (\mathbb{S} \setminus \mathbb{P})^{(1)}$ , and the sets  $\mathbb{P}$  and (again)  $\mathbb{S} \setminus \mathbb{P}$  are not closed, since  $\beta_{(n,k,a)} \in \mathbb{P}$ ,  $\theta_{(k,a)} \in \mathbb{S} \setminus \mathbb{P}$  and  $a + 1 \in \mathbb{P}$ .  $\square$

**Proof of Theorem 2.** (i) Let  $\beta$  be a beta-number with degree  $d$ . If  $d = 1$ , then  $\beta$  is a rational integer,  $\{\beta\} = 0$  and  $\beta \in \mathbb{P}$ . Suppose  $d = 2$ . Since a beta-number has no conjugate, other than  $\beta$  itself, in the interval  $]1, \infty[$  [22], the conjugate  $\gamma$  of a quadratic beta-number  $\beta$ , other than  $\beta$ , belongs to the interval  $] - 2, 1[$ . To show that  $\beta \in \mathbb{S}$ , it suffices to prove that  $\gamma \notin ] - 2, -1[$ .

A short computation gives that each interval of the form  $[n, n + 1]$ , where  $n \in \mathbb{N}$ , contains  $2n - 1$  quadratic Pisot numbers  $\theta$ : the minimal polynomial of  $\theta$  is  $P(x) = x^2 - (n + 1)x + p$ , with  $p \in \{1, \dots, n - 1\}$ , or  $Q(x) = x^2 - nx - p$ , with  $p \in \{1, \dots, n\}$ . If  $P(\theta) = 0$ , then  $\theta$  has a conjugate in  $]0, 1[$  and so  $\theta \in \mathbb{S} \setminus \mathbb{P}$ ; otherwise  $\{\theta\} = \theta - n \equiv \frac{p}{\theta}$  and  $\theta \in \mathbb{P}$ .

Now, assume on the contrary that  $\gamma \in ] - 2, -1[$ . Then,  $\beta + 1$  is a quadratic Pisot number, and so  $\beta$  is a root of one of the irreducible polynomials  $Q(x + 1) = x^2 - (n - 2)x - (n + p - 1)$ , where  $p \in \{1, \dots, n\}$ . If  $\beta$  is simple, then there is  $k \in \mathbb{N}$ , such that

$$\{\beta\} = \beta - \varepsilon_0 \equiv \sum_{j=1}^k \frac{\varepsilon_j}{\beta^j},$$

and so  $Q(x + 1)$  divides, in the ring  $\mathbb{Z}[x]$ , the polynomial  $x^{k+1} - \varepsilon_0 x^k - \dots - \varepsilon_k$ ; thus  $n + p - 1$  divides  $\varepsilon_k$  in  $\mathbb{Z}$ , and this leads to a contradiction, since  $n + p - 1 \geq n$  and  $1 \leq \varepsilon_k \leq \{\beta\} < \beta \leq n$ .

Similarly, if  $\beta$  is not simple, then we have

$$\{\beta\} = \beta - \varepsilon_0 \equiv \sum_{j=1}^{k-1} \frac{\varepsilon_j}{\beta^j} + \sum_{l=0}^{\infty} \sum_{j=k}^{k+q-1} \frac{\varepsilon_j}{\beta^{j+lq}},$$

for some  $q$  and  $k \in \mathbb{N}$  (if  $k$  exists). By the equation  $\beta - \varepsilon_0 = \sum_{j=1}^{k-1} \frac{\varepsilon_j}{\beta^j} + \frac{\beta^q}{\beta^{q-1}} \sum_{j=k}^{k+q-1} \frac{\varepsilon_j}{\beta^j}$ , we obtain a monic polynomial  $R(x)$ , which is a multiple of  $Q(x+1)$  in  $\mathbb{Z}[x]$ , and this relation leads also to a contradiction, because  $|R(0)| = |\varepsilon_{k-1} - \varepsilon_{k+q-1}| \leq [\beta] < n$ . Finally, notice that the cubic case follows from [3].

(ii) Consider the family of polynomials

$$P(x) = P_k(x) := x^3 - kx^2 - (k+1)x + 1,$$

where  $k \in \mathbb{N} \cap [2, \infty[$ . Then, the relations  $P(-2) = -5 - 2k$ ,  $P(-1) = 1 = P(0)$ ,  $P(1) = 1 - 2k$ ,  $P(k) = 1 - k(k+1)$  and  $P(k+1) = 1$ , imply that  $P$  has a root in each one of the following intervals  $] -2, -1[$ ,  $]0, 1[$  and  $]k, k+1[$ , and so  $P$  is irreducible. Let  $\beta = \beta_k$  be the root of  $P$  which belongs to the interval  $]k, k+1[$ . Then,  $\beta \notin \mathbb{S}$ ,  $[\beta] = k$  and  $\{\beta\} = \beta - k = (k+1)/\beta - 1/\beta^2$ . Hence,  $\beta\{\beta\} = k+1 - 1/\beta$ ,  $k < \beta\{\beta\} < k+1$ ,  $\varepsilon_1 = k$ ,  $r_1 = 1 - 1/\beta$ ,  $\beta r_1 = \beta - 1$ ,  $\varepsilon_2 = k - 1$  and  $r_2 = \{\beta\}$ . It follows that  $r_{j+2} = r_j$ ,  $\varepsilon_{2j-1} = k$  and  $\varepsilon_{2j} = k - 1$ ,  $\forall j \in \mathbb{N}$ ; thus

$$\{\beta\} \equiv \sum_{j \geq 1} \left( \frac{k}{\beta^{2j-1}} + \frac{k-1}{\beta^{2j}} \right)$$

and  $\beta$  is a beta-number.

(iii) Let  $d \in \mathbb{N} \cap [4, \infty[$ . First, suppose that  $d$  is even, and consider the polynomial

$$P(x) = P_d(x) := x^d - px^{d-1} - px^{d-2} - p,$$

where  $p$  is prime. Then,  $P$  is  $p$ -Eisenstein,  $P(p) = -p^{d-1} - p$ ,  $P(p+1) = (p+1)^{d-2} - p$ , and so  $P$  is the minimal polynomial of a real number  $\beta = \beta_d \in ]p, p+1[$ . Furthermore, we have  $[\beta] = p$ ,  $\{\beta\} = \beta - p = p/\beta + p/\beta^{d-1}$ ,  $\beta\{\beta\} = p + p/\beta^{d-2}$ ,  $(r_1, \varepsilon_1) = (p/\beta^{d-2}, p)$ ,  $(r_j, \varepsilon_j) = (p/\beta^{d-1-j}, 0)$ ,  $\forall j \in \{2, \dots, d-2\}$ , and  $(r_{d-1}, \varepsilon_{d-1}) = (0, p)$ . Hence,  $\{\beta\} \equiv \frac{p}{\beta} + \frac{p}{\beta^{d-1}}$  and  $\beta$  is a simple beta-number which is not a Pisot number, since  $\beta$  has a conjugate in  $] -2, -1[$  (we have  $P(-1) = 1 - p < 0$  and  $P(-2) = 2^d + 2^{d-2}p - p > 0$ ).

Now, assume that  $d$  is odd, and consider

$$P(x) = P_d(x) := x^d - 2px^{d-1} - 2px^{d-2} - 2px^{d-4} - p,$$

where  $p$  is prime. Similarly as for the case where  $d$  is even, we have that  $P$  is  $p$ -Eisenstein,  $P(2p) < 0$ ,  $P(2p+1) > 0$ ,  $P(-1) > 0$ ,  $P(-2) < 0$ , and so  $P$  is the minimal polynomial of an algebraic integer  $\beta = \beta_d \in ]2p, 2p+1[$ , having a conjugate in the interval  $] -2, -1[$ . Moreover,  $[\beta] = 2p$ ,  $\{\beta\} = \beta - 2p$  and an easy calculation gives  $\{\beta\} \equiv \frac{2p}{\beta} + \frac{2p}{\beta^3} + \frac{p}{\beta^{d-1}}$ . Hence,  $\beta$  is a simple beta-number which does not belong to  $\mathbb{S}$ .  $\square$

**Proof of Theorem 3.** To simplify the notation, set  $P_k := P_{(n_k, \dots, n_0; a)}$ ,  $\forall k \in \{0, \dots, a-2\}$ . It is clear that  $P_1(1) = 2 - a$ ,  $P_1(a-1) = 1 - (a-1)^{n_1} \leq -1$  and  $P_1(a) = 1$ . Assume that we have  $P_{k-1}(1) = k - a$ ,  $P_{k-1}(a-1) \leq -1$  and  $P_{k-1}(a) \geq 1$ , where  $k \geq 2$ . Then, (1) gives immediately  $P_k(1) = k+1 - a$ ,  $P_k(a-1) \leq -1$  and  $P_k(a) \geq 1$ . It follows by induction that  $P_k(1) < 0$ ,  $P_k(a-1) < 0$  and  $P_k(a) > 0$ . Hence, if  $k \in \{1, \dots, a-2\}$ , then the polynomial

$P_k$  has two real roots, one  $\rho_k = \rho_{(n_k, \dots, n_0; a)} \in ]0, 1[$  (recall that  $P_k(0) = 1$ ), and the other  $\theta_k = \theta_{(n_k, \dots, n_0; a)} \in ]a - 1, a[$ . Furthermore, the equalities  $P_k(\rho_{k-1}) = P_k(\theta_{k-1}) = 1$ , where  $k \in \{2, \dots, a - 2\}$ , obtained from (1), yield  $\rho_k \in ]\rho_{k-1}, 1[$  and  $\theta_k \in ]a - 1, \theta_{k-1}[$ ; thus

$$a - 1 < \theta_{a-2} < \dots < \theta_0 := \theta_{(n_0; a)} = a. \quad (2)$$

It is clear that  $\theta_0 \in \mathbb{X}$ . For  $k \in \{1, \dots, a - 2\}$ , we shall prove that  $\rho_k$  is a conjugate of  $\theta_k$ , and the other conjugates of  $\theta_k$  are of modulus less than  $\rho_k$ . Let  $z$  be a complex number with modulus 1. Then,  $|P_0(z)| \geq a - 1$ . Moreover, if  $|P_{k-1}(z)| \geq a - k$  for some  $k \geq 1$ , then by (1) we obtain  $|P_k(z)| \geq |P_{k-1}(z)| - 1 \geq a - (k + 1)$ ,  $|P_k(z)| \geq 1$  for all  $k \in \{0, \dots, a - 2\}$ , and

$$|P_k(z)| > 1, \quad \forall k \leq a - 3. \quad (3)$$

Now, a simple induction shows that  $\theta_k \in \mathbb{S}$  and  $P_k$  is the minimal polynomial of  $\theta_k$ . Indeed, this last proposition is trivially true for  $k = 0$ . Assume that  $P_{k-1}$  is the minimal polynomial of the Pisot number  $\theta_{k-1}$ , where  $k \in \{1, \dots, a - 2\}$ . Then, using Rouché's theorem, the relation (1) together with (3) give that  $P_k$  has  $n_k + \deg(P_{k-1}) - 1 = \deg(P_k) - 1$  roots with modulus less than 1, and the result follows immediately, since  $P_k$  has one root with modulus  $> 1$ , namely  $\theta_k$ .

To complete the proof of the relation:  $\theta_k \in \mathbb{X}$ , consider a conjugate  $\alpha$  of  $\theta_k$ , with modulus less than 1, where  $k \geq 1$ . Then, writing (1) explicitly, we obtain

$$\begin{aligned} \alpha^{n_k + \dots + n_1 + 1} - a\alpha^{n_k + \dots + n_1} + \alpha^{n_k + \dots + n_2} + \dots + \alpha^{n_k} + 1 &= 0, \\ |\alpha|^{n_k + \dots + n_1 + 1} + |\alpha|^{n_k + \dots + n_2} + \dots + |\alpha|^{n_k} + 1 &\geq a |\alpha|^{n_k + \dots + n_1}, \end{aligned}$$

and so

$$P_k(|\alpha|) \geq 0.$$

Consequently, to show the inequality  $|\alpha| \leq \rho_k$ , it suffices to verify that  $P_k(t) < 0$ ,  $\forall t \in ]\rho_k, 1[$ . In fact, we use again an induction on  $k$ , to show that  $P'_k(t) < 0$  and  $P_k(t) < 0$ ,  $\forall t \in ]\rho_k, 1[$ . Clearly,  $P'_1(t) = (n + 1)t^{n-1}(t - nr/(n + 1)) < 0$ ,  $\forall t \in ]0, nr/(n + 1)[$ , and so the quantities  $P'_1(t)$  and  $P_1(t)$  are negative when  $t \in ]\rho_1, 1[$ , since  $]\rho_1, 1[ \subset ]0, nr/(n + 1)[$  and  $P_1(\rho_1) = 0$ . Suppose that  $P_{k-1}(t) < 0$  and  $P'_{k-1}(t) < 0$ , where  $t \in ]\rho_{k-1}, 1[$  and  $k \in \{2, \dots, a - 2\}$ . Then, the relation  $P'_k(t) = t^{n_k-1}(n_k P_{k-1}(t) + t P'_{k-1}(t))$ , gives  $P'_k(t) < 0$  for  $t \in ]\rho_k, 1[$ , since  $]\rho_k, 1[ \subset ]\rho_{k-1}, 1[$ , and so

$$P_k(t) < P_k(\rho_k) = 0,$$

$\forall t \in ]\rho_k, 1[$ . Hence,  $|\alpha| \leq \rho_k$ , and by the above mentioned result of Smyth [21] we deduce that  $|\alpha| < \rho_k$ ; thus  $\theta_k \in \mathbb{X}$ .

Now, the relation (1) together with (2) yield

$$\lim_{n_k \rightarrow \infty} P_{k-1}(\theta_k) = 0 \quad \text{for } k \geq 2,$$

since  $|P_{k-1}(\theta_k)| = 1/\theta_k^{n_k} < 1/(a - 1)^{n_k}$ ; thus

$$\lim_{n_k \rightarrow \infty} \theta_k = \theta_{k-1}, \quad (4)$$

because  $\theta_{k-1}$  is the unique root with modulus greater than 1 of the polynomial  $P_{k-1}$ .  $\square$

**Proof of the Corollary.** With the same notation as in the proof above, the relation (4) gives immediately  $\theta_{a-2-k} \in \mathbb{X}^{(k)}$ ,  $\forall k \in \{1, \dots, a - 3\}$ . The inequality  $|\theta_1 - a| = \frac{1}{\theta_1^{n_1}} < \frac{1}{(a-1)^{n_1}}$ , implies also  $\lim_{n_1 \rightarrow \infty} \theta_1 = a$ , and so  $\theta_0 \in \mathbb{X}^{(a-2)}$ .  $\square$

**Remark.** If we consider the polynomials  $x^n(x - r) + k$ , where the rational integers  $r$  and  $k$  satisfy  $r \geq k + 2 \geq 4$ , then by the same way as in the proof of Theorem 3, we obtain families of strong Pisot which are not units.

**Proof of Theorem 4.** To show Theorem 4(i), we shall exhibit a real model set  $\Lambda_\varepsilon$ , such that each element of  $\Lambda_\varepsilon \cap ]1, \infty[$  is an  $\varepsilon$ -strong Pisot number generating  $K$ , and after this, we conclude by a result of Meyer [12], which says that a real model set is a relatively dense subset of  $\mathbb{R}$  (for more details on model sets, see also [13] or [26]). It is clear that Theorem 4(i) is true for  $d = 1$ , since the set of  $\varepsilon$ -strong Pisot number generating  $\mathbb{Q}$  is  $\mathbb{N} \cap [2, \infty[$ . Now, suppose that  $d \geq 2$ , and let  $\sigma_1, \dots, \sigma_d$  be the distinct embeddings of  $K$  in  $\mathbb{C}$ , where the first  $r$  ones are real,  $\sigma_1$  is the identity of  $K$ ,  $r \geq 2$ , and  $\sigma_{j+(d-r)/2}(\alpha)$  is the complex conjugate  $\overline{\sigma_j(\alpha)}$  of  $\sigma_j(\alpha)$ , where  $j \in \{r+1, \dots, r+(d-r)/2\}$  and  $\alpha \in K$ . For a fixed base  $\{\omega_1, \dots, \omega_d\}$  of the ring of the integers of  $K$ , consider the linear forms  $l_1, \dots, l_d$  defined on  $\mathbb{R}^d$  by the relations

$$l_j(x_1, \dots, x_d) = \sum_{k=1}^d x_k \sigma_j(\omega_k)$$

when  $j \in \{1, 2, \dots, r\}$ , and

$$l_j(x_1, \dots, x_d) = \sum_{k=1}^d x_k (\sigma_j(\omega_k) + \sigma_{j+(d-r)/2}(\omega_k))/2$$

and

$$l_{j+(d-r)/2}(x_1, \dots, x_d) = \sum_{k=1}^d x_k (\sigma_j(\omega_k) - \sigma_{j+(d-r)/2}(\omega_k))/2i,$$

where  $i^2 = -1$ , when  $j \in \{r+1, \dots, r+(d-r)/2\}$ . Consider also the subset

$$\Omega_\varepsilon = ]\varepsilon/2, \varepsilon[ \times ]0, \varepsilon/2[^{r-2} \times \bigcap_{j=1}^{(d-r)/2} \{(y_1, \dots, y_{d-r}) \in \mathbb{R}^{d-r} \mid y_j^2 + y_{j+(d-r)/2}^2 < \varepsilon^2/4\}$$

of the Euclidean space  $\mathbb{R}^{d-1}$ . Then, the set

$$\Lambda_\varepsilon = \left\{ \sum_{k=1}^d p_k \omega_k \mid (p_1, \dots, p_d) \in \mathbb{Z}^d, (l_2(p_1, \dots, p_d), \dots, l_d(p_1, \dots, p_d)) \in \Omega_\varepsilon \right\},$$

is a real model set [12]. It is clear that  $\Lambda_\varepsilon$  is contained in the ring of the integers of  $K$ . Let  $\alpha \in \Lambda_\varepsilon$ . Then, the relation  $\prod_{j=1}^d |\sigma_j(\alpha)| \geq 1$ , gives  $|\alpha| > 1$ , and so the conjugates of  $\alpha$  are exactly the numbers  $\sigma_1(\alpha), \dots, \sigma_d(\alpha)$ . In particular, if  $\alpha > 0$ , then  $\alpha > 1$ ,  $\varepsilon/2 < \sigma_2(\alpha) < \varepsilon$ , and  $|\sigma_j(\alpha)| < \varepsilon/2$  for  $j \geq 3$ ; thus  $\alpha$  is an  $\varepsilon$ -strong Pisot number, and this ends the proof of Theorem 4(i).

Now, suppose that  $K$  is generated by a Salem number. Then,  $r = 2$  and  $d \geq 4$ . Let  $\Lambda$  be a model set defined by

$$\Lambda = \left\{ \sum_{k=1}^d p_k \omega_k \mid (p_1, \dots, p_d) \in \mathbb{Z}^d, (l_2(p_1, \dots, p_d), \dots, l_d(p_1, \dots, p_d)) \in \Omega \right\},$$

where  $\Omega = [-1, 1] \times \bigcap_{j=1}^{(d-2)/2} \{(y_1, y_2, \dots, y_{d-2}) \in \mathbb{R}^{d-2}, y_j^2 + y_{j+d-2}^2 \leq 1\}$ . Then, a simple computation shows that  $\alpha \in \Lambda$  if and only if  $\alpha$  is an integer of  $K$  and  $|\sigma_j(\alpha)| \leq 1$  for each



$j \in \{2, 3, \dots, d\}$ ; thus Salem numbers generating  $K$  belong to  $\Lambda$ . After this we conclude, similarly as in the proof of Theorem 2 of [25], by using Proposition 7.9 of [13], which asserts that there is a finite subset  $F_\varepsilon$  of  $K$  such that  $\Lambda \subset \Lambda_\varepsilon + F$ .  $\square$

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